# THE ELIMINATION METHOD OF FOURIER-MOTZKIN IN LINEAR PROGRAMMING

*Manolis Kritikos* [1*]
*Dimitrios Kallivokas* [2]

## ABSTRACT

*The paper applies the elimination method of Fourier-Motzkin in a production problem in Linear Programming. Following the Fourier-Motzkin elimination method, we successively eliminate variables of the model that solve a linear programming problem until its final solution. The method helps us to find the optimum solution of a linear programming problem without using Linear Programming methodology.*

**KEYWORDS:** *elimination method of Fourier-Motzkin, decision making, optimization, linear programming, system of inequalities.*

## INTRODUCTION

The following problem is a simple production problem (G. Prastacos, 2008) of a business that produces two products, named A and B. Those two products are processed by two departments of the enterprise, respectively named T1 and T2, in order to be produced. The available operating hours of the two processing departments per month are limited and are 2100 and 1800 hours respectively. Furthermore, the production time of each product type is different in each department and is given in Table 1 below.

Table 1: Employment of the departments per unit of output

| PRODUCT | DEPT T1 | DEPT T2 |
|---|---|---|
| A | 2 | 3 |
| B | 3 | 2 |

In our problem, there are additional restrictions concerning claims about customer satisfaction and the limited capacity of storage areas which results in the production of only 400 units of product A and in the production of at least 300 units of product B per month (M. Kritikos and G. Ioannou, 2013). The enterprise has to determine the number of products A and B that should be produced in order to optimize the total profit from the disposal of these 2 products. The operational profit from the sale of products A and B is

[1*] corresponding author, Management Sciences Laboratory, Athens University of Economics and Business, 47A Evelpidon and 33 Lefkados St. 8th Floor, Athens 113-62, Greece, kmn@aueb.gr
[2] Technological Educational Institution of Athens, dimkalliv@yahoo.gr

respectively 5 and 3 credit units. In order to delineate the set of feasible solutions of the problem of the production problem we will use the following inequalities arising from the problem data: Let A be the number of products A that we produce and B the number of products B, respectively. Because every product of type A requires to be processed for two hours in department T1 and every product of type B requires to be processed for three hours in department T1 and the total availability is 2100 hours, we will apply the following inequality in the problem: (2 hours per product A) x (number of products A) + (3 hours per product B) x (number of products B) ≤ total hours available, namely:

$$2A + 3B \leq 2100 \tag{1}$$

Similarly, the inequality in relation to the operation of department T2 would be:

$$3A + 2B \leq 1800 \tag{2}$$

Additionally, because the production should include only 400 units of product A and at least 300 units of product B, the inequalities expressing respectively the restrictions are $A \leq 400$ (3) and $B \geq 300$ (4). Of course, it is obvious that the restrictions of nonnegative output have to be applied, that is $A \geq 0$ (5) and $B \geq 0$ (6).

## FOURIER-MOTZKIN ELIMINATION METHOD

We apply the Fourier-Motzkin elimination method (Dantzig, 1963) in order to determine the optimal solution of the problem. We suppose that we earn 5 and 3 credit points from the sale of products A and B, respectively. In this case our profit occurs from the function z = 5xA +3xB (7). Afterwards we search for a solution which on the one hand, satisfies both (1), (2), (3), (4), (5) and (6) inequalities whereas on the other hand, gives to z its largest credit that does not exceed $5A+3B$ that is, the solution of the problem occurs from the solution of the following system of inequalities:

$$2A+3B \leq 2100 \tag{1}$$

$$3A+2B \leq 1800 \tag{2}$$

$$A \leq 400 \tag{3}$$

$$B \geq 300 \tag{4}$$

$$A \geq 0 \tag{5}$$

$$B \geq 0 \tag{6}$$

$$z \leq 5A+3B \tag{7}$$

Following the Fourier-Motzkin elimination method (Dantzig, 1963) for the solution of the above system, we initially eliminate variable A. For this reason, the system could be written as:

$$A \leq 1050 - \frac{3}{2}B \tag{1}$$

$$A \leq 600 - \frac{2}{3}B \tag{2}$$

$$A \leq 400 \tag{3}$$

$$B \geq 300 \tag{4}$$

$$A \geq 0 \tag{5}$$

$$B \geq 0 \tag{6}$$

$$A \geq \frac{z}{5} - \frac{3}{2}B \tag{7}$$

It should be observed that the above inequalities relating to variable A, can be grouped: into those in which A is larger than a linear relationship, in those that A is smaller and those that do not exhibit the variable A. Of course, it is evident that the values of the first group are smaller than the values of the second group. Thus, we get the following system:

$$\frac{z}{5} - \frac{3}{5}B \leq 1050 - \frac{3}{2}B \tag{1}$$

$$\frac{z}{5} - \frac{3}{5}B \leq 600 - \frac{2}{3}B \tag{2}$$

$$\frac{z}{5} - \frac{3}{5}B \leq 400 \tag{3}$$

$$1050 - \frac{3}{2}B \geq 0 \tag{4}$$

$$600 - \frac{2}{3}B \geq 0 \tag{5}$$

$$400 \geq 0 \tag{6}$$

$$B \geq 0 \tag{7}$$

$$B \geq 300 \tag{8}$$

Repeating the process in order to eliminate the variable B gives us the following system of inequalities:

$$B \le \frac{10}{9}(1050 - \frac{z}{5}) \tag{1}$$

$$B \le 15(600 - \frac{z}{5}) \tag{2}$$

$$B \ge \frac{5}{3}(\frac{z}{5} - 400) \tag{3}$$

$$B \le \frac{2}{3}1050 \tag{4}$$

$$B \le \frac{3}{2}600 \tag{5}$$

$$B \ge 0 \tag{6}$$

$$B \ge 300 \tag{7}$$

Grouping the inequalities as in case of variable A results in inequalities of variable z whose solution results in the value of z. For example the combination of (1) and (3) gives us the inequality:

$$(\frac{5z}{15} - \frac{2000}{3}) \le (10500 - \frac{10z}{5}) / 9 \Rightarrow z \le 3300 \,.$$

Similarly, the other inequalities of the method occur:

$$\frac{5}{3}(\frac{z}{5} - 400) \le 15(600 - \frac{z}{5}) \Rightarrow z \le 2900$$

$$\frac{5}{3}(\frac{2}{5} - 400) \le \frac{2}{3}1050 \Rightarrow z \le 4100$$

$$\frac{5}{3}(\frac{2}{5} - 400) \le \frac{3}{2}600 \Rightarrow z \le 4700$$

$$10(1050 - \frac{z}{5}) / 9 \ge 0 \Rightarrow z \le 5250$$

$$10(1050 - \frac{z}{5}) / 9 \ge 300 \Rightarrow z \le 3900$$

It is evident from the above solutions that the solution for z is z = 2900. We put z=2900 in the inequalities system before the elimination of B so that B = 300. Then we set the values z = 2900 and B = 300 in the original inequalities system so that A=400. Namely, we have z = 2900, A = 400 and B = 300. We confirm the solution using the PHPSimplex tool http://www.phpsimplex.com/simplex/simplex.htm?l=en).

## CONCLUSION

The paper applies the elimination method of Fourier-Motzkin in Linear Programming. Working through a system of inequalities showed the usefulness of mathematics in an application of a simple operational problem. The method helps us to find the optimum solution of a production problem without using linear programming.

## REFERENCES

[1] G. B. Dantzig., (1963) Linear Programming and Extensions, Princeton University Press

[2] M. N. Kritikos and G. Ioannou, (2013) The heterogeneous fleet vehicle routing problem with overloads and time windows, International Journal of Production Economics, 144, 68-75

[3] G. P. Prastacos, (2008), Managerial Decision Making Theory and Practice, Tsinghua University Press

[4] http://www.phpsimplex.com/simplex/simplex.htm?l=en